

# NON-ERGODIC BANACH SPACES ARE NEAR HILBERT

W. CUELLAR CARRERA

**ABSTRACT.** We prove that a non ergodic Banach space must be near Hilbert. In particular,  $\ell_p$  ( $2 < p < \infty$ ) is ergodic. This reinforces the conjecture that  $\ell_2$  is the only non ergodic Banach space. As an application of our criterion for ergodicity, we prove that there is no separable Banach space which is complementably universal for the class of all subspaces of  $\ell_p$ , for  $1 \leq p < 2$ . This solves a question left open by W. B. Johnson and A. Szankowski in 1976.

## 1. INTRODUCTION

The solution of Gowers [20] and Komorowski–Tomczak–Jaegermann [30] to the homogeneous Banach space problem, provides that every Banach space having only one equivalence class for the relation of isomorphism between its infinite dimensional subspaces must be isomorphic to  $\ell_2$ . G. Godefroy formulated the question about the number of non isomorphic subspaces of a Banach space  $X$  not isomorphic to  $\ell_2$ . This question was studied, in the context of descriptive set theory, by V. Ferenczi and C. Rosendal [16] who introduced the notion of *ergodic Banach space* to study the classification of the relative complexity of the isomorphism relation between the subspaces of a separable Banach space.

Our general reference for descriptive set theory will be the book [29]. A *Polish space* is a separable topological space which admits a compatible complete metric. The Borel sets of a Polish space is the  $\sigma$ -algebra generated by the open sets. A set  $X$  equipped with a  $\sigma$ -algebra is called a *Borel standard space* if there exists a Polish topology on  $X$  for which that  $\sigma$ -algebra arises as the collection of Borel subsets of  $X$ . A function between two Borel standard spaces  $f : X \rightarrow Y$  is said to be Borel, if  $f^{-1}(B)$  is Borel in  $X$ , for every Borel subset  $B \subseteq Y$ .

Given a Polish space  $X$ , let  $\mathcal{F}(X)$  be the collection of all closed subsets of  $X$ . The  $\sigma$ -algebra on  $\mathcal{F}(X)$  generated by

$$A_U = \{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\},$$

where  $U$  is an open subset of  $X$ , is called the *Effros Borel structure* on the closed subsets of  $X$ . It is not hard to see that  $\mathcal{F}(X)$  equipped with this Borel structure is a Borel standard space.  $\mathcal{SB}(X)$  denotes the collection of infinite dimensional linear subspaces  $Y \in \mathcal{F}(X)$  equipped with the relative Effros Borel structure. This framework allows us to identify every class of subspaces of a Banach space  $X$  with a subset of  $\mathcal{SB}(X)$  in which its

---

2010 *Mathematics Subject Classification.* Primary 46B20, 46B03, Secondary 03E15.

The author was supported by FAPESP grant 2014/25900-7.

complexity can be measured. Since  $C(2^{\mathbb{N}})$  is universal for all separable Banach spaces, we have that properties of separable Banach spaces becomes sets in  $\mathcal{SB}(C(2^{\mathbb{N}}))$ . In [4] it was proved that the relation of isomorphism between separable Banach spaces is an analytic and not borelian subset of  $\mathcal{SB}(C(2^{\mathbb{N}}))^2$ .

The central notion to study the complexity of analytic and Borel equivalence relations on Borel standard spaces is the concept of *Borel reducibility*, which originated from the works H. Friedman and L. Stanley [19] and independently from the works of L. A. Harrington, A. S. Kechris and A. Louveau [21].

**Definition 1.** *Let  $R$  and  $S$  be two Borel equivalence relations on Borel standard spaces  $X$  and  $Y$ , respectively. One says that  $R$  is Borel reducible to  $S$ , (denoted by  $R \leq_B S$ ) if there exists a Borel function  $\phi : X \rightarrow Y$  such that*

$$xRy \iff \phi(x)S\phi(y),$$

*for all  $x, y \in X$ . The relation  $R$  is Borel bireducible to  $S$ , (denoted by  $R \sim_B S$ ), whenever both  $R \leq_B S$  and  $S \leq_B R$  hold.*

This can be interpreted as that the equivalence relation  $R$  is classified by a Borel assignment of invariants provided by equivalence classes for  $S$ . Observe that a Borel reduction induces an embedding from the quotient space  $X/R$  to  $Y/S$ , so  $X/R$  has less than or equal cardinality that of  $Y/S$ .

Ferenczi, Louveau and Rosendal [15] proved that the relation of isomorphism between separable Banach spaces is a complete analytic equivalence relation, i.e., that any analytic equivalence relation Borel reduces to it.

For  $X$  a Polish space, let  $\text{id}(X)$  be the identity relation on the space  $X$ . Since any two standard Borel spaces with the same cardinality are Borel isomorphic, it follows that for any uncountable  $X$ ,

$$\text{id}(X) \sim_B \text{id}(\mathbb{R}).$$

Among the uncountable Borel equivalence relations, the simplest is  $\text{id}(\mathbb{R})$ . In fact, it was proved by Silver [38] that given a Borel equivalence relation  $(X, R)$ , either it has countable many classes of equivalence, or  $\text{id}(\mathbb{R})$  is Borel reducible to  $(X, R)$ . An equivalence relation admitting the reals as a complete invariant is called *smooth*, that is, when it is reducible to  $\text{id}(\mathbb{R})$ .

The simplest example of a non-smooth equivalence relation is the relation of eventual agreement  $E_0$  on  $2^{\mathbb{N}}$ , i.e., for  $x, y \in 2^{\mathbb{N}}$ ,

$$xE_0y \iff (\exists N \in \mathbb{N})(x(n) = y(n), n \geq N).$$

Harrington, Kechris, and Louveau [21] proved that  $E_0$  is minimal among non-smooth Borel equivalence relations with respect to  $\leq_B$ .

The following notion measures the complexity of the relation of isomorphism between subspaces of a separable Banach space and was introduced by Ferenczi and Rosendal [16].

**Definition 2.** *A separable Banach space  $X$  is ergodic if*

$$(2^{\mathbb{N}}, E_0) \leq_B (\mathcal{SB}(X), \simeq).$$

It follows that an ergodic Banach space has at least  $2^{\mathbb{N}}$  non-isomorphic subspaces and the equivalence relation of isomorphism between its subspaces is non-smooth.

Rosendal [37] notices that every hereditarily indecomposable (H.I) Banach space (i.e., a space in which no closed infinite dimensional subspace can be written as the direct sum of two closed infinite dimensional subspaces) is ergodic. By Gowers dichotomy [20], every Banach space contains an H.I subspace or an unconditional basic sequence. Since every Banach space containing an ergodic subspace must be ergodic, one can approach the study of ergodicity by first restricting to spaces with unconditional basis.

Ferenczi and Rosendal [16] proved that a non ergodic Banach space  $X$  with unconditional basis satisfies some regularity properties like being isomorphic to its square and to its hyperplanes, and more generally must be isomorphic to  $X \oplus Y$  for any subspace  $Y$  of  $X$  generated by a subsequence of the basis. It was conjectured in [16] that every separable Banach space not isomorphic to  $\ell_2$  must be ergodic.

Dilworth, Ferenczi, Kutzarova and Odell [12] proved that every Banach space  $X$  with a strongly asymptotic  $\ell_p$  basis ( $1 \leq p \leq \infty$ ) not equivalent to the unit vector basis of  $\ell_p$  (or  $c_0$  if  $p = \infty$ ) is ergodic. This result was generalized by R. Anisca [2], who constructed explicit Borel reductions to prove that every separable asymptotically Hilbertian space (and therefore every weak Hilbert space) not isomorphic to  $\ell_2$  is ergodic.

Recall that a Banach space  $X$  is called *(complementably) minimal* (notions due to Pełczyński and Rosenthal, respectively) if every infinite-dimensional closed subspace  $Y$  of  $X$  contains a (complemented) subspace  $Z$  isomorphic to  $X$ . Clearly, every (complemented) subspace of a (complementably) minimal space is also a (complementably) minimal space. Ferenczi [13] proved that a separable Banach space without minimal subspaces must be ergodic. Hence, the conjecture in [16] is related to the following problem: *Is every minimal Banach space not isomorphic to  $\ell_2$  ergodic?*

It is well known that  $c_0$  and  $\ell_p$  ( $1 \leq p < \infty$ ) are complementably minimal spaces, while the dual of the Tsirelson space  $T^*$  is an example of minimal but not complementably minimal space [7]. The first example of a complementably minimal space other than  $c_0$  and the  $\ell_p$ 's is the Schlumprecht space and its dual [40]. The list of minimal spaces known so far is completed with the family of Schlumprecht type spaces and their duals constructed by complex interpolation methods in [8], and every infinite dimensional closed subspace of each of the above. For classical spaces, it was proved in [14] that  $c_0$  and  $\ell_p$  for  $1 \leq p < 2$  are ergodic. Rosendal [36] proved that the dual of the Tsirelson space is ergodic. In this work we prove ergodicity for a general family of Banach spaces including all the other minimal spaces not isomorphic to  $\ell_2$  listed above. More specifically, given a Banach space  $X$ , let

$$\begin{aligned} p(X) &= \sup\{p : X \text{ has type } p\}, \\ q(X) &= \inf\{q : X \text{ has cotype } q\}. \end{aligned}$$

Recall that a Banach space  $X$  is said to be *near Hilbert* when  $p(X) = q(X) = 2$ . We give a criterion for ergodicity which together with the Johnson and Szankowski construction of subspaces without the approximation property allows to prove that a non-ergodic Banach space must be near Hilbert. In particular, we solve the question of [16] about the ergodicity

of the  $\ell_p$  spaces, for  $p > 2$ . We also prove that the family of Schlumprecht type spaces and its dual are not near Hilbert, and therefore they are ergodic spaces.

Finally, as an application of the criterion for ergodicity, we prove that for every non near Hilbert space  $X$  there does not exist separable Banach space which is complementably universal for the class of all subspaces of  $X$ . In particular, this is true for  $X = \ell_p$ ,  $p \neq 2$ . This solves a problem left open by Johnson and Szankowski in their 1976 paper [25] and mentioned again in [24]. (Johnson and Szankowski verified the case  $2 < p < \infty$  in [25])

## 2. CRITERION FOR ERGODICITY

A Banach space  $X$  has the approximation property (AP) if the identity operator on  $X$  can be approximated uniformly on compact subsets of  $X$  by linear operators of finite rank. The Banach space  $X$  is said to have the bounded approximation property (BAP) if there exists  $\lambda > 0$  such that the finite rank operator  $T$  in the definition of AP can be taken with norm  $\|T\| \leq \lambda$ . In 1973, Enflo [11] presented the first example of Banach space without the AP and therefore without a Schauder basis. Enflo's construction was simplified by Davie [10] who used probabilistic methods to construct such examples inside  $\ell_p$ -spaces ( $2 < p \leq \infty$ ). Later in 1978, Szankowski [39] proved that the other range of  $\ell_p$ -spaces ( $1 \leq p < 2$ ) also have subspaces failing AP. The criterion we introduce to study ergodicity in Banach spaces is based on a criterion introduced by Enflo, and used in the works of Davie and Szankowski, to prove that a space fails the AP.

We first introduce some notation used throughout the paper. For every  $n \in \mathbb{N}$ , denote by  $I_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$ . Given a Banach space  $X$  and sequences of vectors  $(z_{n,\epsilon})_{n \in \mathbb{N}}$  in  $X$ ,  $(z_{n,\epsilon}^*)_{n \in \mathbb{N}}$  in  $X^*$ , ( $\epsilon = 0, 1$ ), we denote by  $Z = \overline{\text{span}}\{z_{j,\epsilon} : j \in \mathbb{N}, \epsilon = 0, 1\}$  and we shall consider for every  $t \in 2^{\mathbb{N}}$  the closed subspace

$$X_t = \overline{\text{span}}\{z_{j,t(n)} : j \in I_n, n = 1, 2, 3, \dots\}.$$

If  $T : X_t \rightarrow Z$  is a bounded and linear operator we define the  $n$ -trace of  $T$  as

$$\beta_t^n(T) = 2^{-n} \sum_{j \in I_n} z_{j,t(n)}^* T(z_{j,t(n)}).$$

**Definition 3.** A Banach space  $X$  satisfies the Cantorized-Enflo criterion if there exist bounded sequences of vectors  $(z_{n,\epsilon})_{n \in \mathbb{N}}$  in  $X$ ,  $(z_{n,\epsilon}^*)_{n \in \mathbb{N}}$  in  $X^*$  ( $\epsilon = 0, 1$ ) and a sequence of real scalars  $(\alpha_n)_n$  such that

- (1)  $z_{i,\epsilon}^*(z_{j,\tau}) = \delta_{ij}\delta_{\epsilon\tau}$  for all  $i, j \in \mathbb{N}$  and  $\epsilon, \tau = 0, 1$ .
- (2) For every  $t, s \in 2^{\mathbb{N}}$  and every operator  $T : X_t \rightarrow X_s$

$$|\beta_t^n(T) - \beta_t^{n-1}(T)| \leq \alpha_n \|T\|$$

- (3)  $\sum_n \alpha_n < \infty$ .

Recall that a subset of a topological space is said to be meagre if it is the countable union of nowhere dense subsets (sets whose closure has empty interior). An equivalence relation on a standard Borel space  $X$  is said to be meagre if it is a meagre subset of  $X^2$ .

Let  $t \in 2^{\mathbb{N}}$  and  $n \in \mathbb{N}$ . We denote by  $t/n = \{k \leq n : t(k) = 1\}$ . The  $E'_0$  equivalence relation on  $2^{\mathbb{N}}$  is defined as

$$xE'_0y \iff \exists n(|t/n| = |s/n|) \wedge (t(k) = s(k), k \geq n).$$

$E'_0$  is a refinement of  $E_0$ , that is  $E'_0 \subseteq E_0$ . In connection with Borel reducibility ordering we shall use the following result from Rosendal [37].

**Proposition 4.** [37, Prop. 15] *Let  $E$  be a meagre equivalence relation on  $2^{\mathbb{N}}$  containing  $E'_0$ . Then  $E_0 \leq_B E$ .*

Recall that a Banach space  $X$  is said to be *complementably universal* for a family  $\mathcal{A}$  of Banach spaces if every space in  $\mathcal{A}$  is isomorphic to a complemented subspace of  $X$ . In [25] Johnson and Szankowski proved that there is no separable Banach space which is complementably universal for the class  $\mathcal{A}_p$  of all subspaces of  $\ell_p$ ,  $2 < p < \infty$ . As observed by W. B. Johnson [23], it follows that a complementably universal Banach space for the class  $\mathcal{A}_p$  ( $2 < p < \infty$ ) must have density character at least the continuum., where the density character of a topological space  $X$  is the least cardinality of a dense subset of  $X$ . In particular, this shows that the family of non isomorphic subspaces of  $\ell_p$ , for ( $2 < p < \infty$ ) has the cardinality of the continuum. We use the ideas of the proof in [25] (see also [26]) to establish a criterion for ergodic Banach spaces.

**Lemma 5.** *Let  $X$  be a Banach space satisfying the Cantorized-Enflo criterion, and  $\Gamma \subseteq 2^{\mathbb{N}}$ . Then every Banach space which is complementably universal for the family  $\{X_t\}_{t \in \Gamma}$  has density character at least cardinality of  $\Gamma$ .*

*Proof.* Let  $X$  be a Banach space satisfying the Cantorized-Enflo criterion and consider sequences  $(z_{n,\epsilon})_{n \in \mathbb{N}}$  on  $X$ ,  $(z_{n,\epsilon}^*)_{n \in \mathbb{N}}$  on  $X^*$ , ( $\epsilon = 0, 1$ ), and real scalars  $(\alpha_n)_n$  as in Definition 3. Suppose that there is  $\Gamma \subseteq 2^{\mathbb{N}}$  uncountable and a separable Banach space  $W$  such that for every  $t \in \Gamma$ ,  $X_t$  is isomorphic to a complemented subspace of  $W$ . We may assume that for some  $\lambda > 0$  there exist an embedding  $T_t : X_t \rightarrow W$  and a projection onto  $P_t : X_t \rightarrow T_t X_t$ , for every  $t \in \Gamma$ , such that  $\|P_t\| \leq \lambda$  and

$$\|x\| \leq \|T_t x\| \leq \lambda \|x\| \text{ for every } x \in X_t.$$

Take  $\delta > 0$ , it follows by conditions (2) and (3) in Definition 3, that there exists  $k = k(\delta)$  such that for every  $m > k$

$$\left| \beta_t^m(T) - \beta_t^k(T) \right| \leq \delta \|T\|,$$

for every  $t, s \in 2^{\mathbb{N}}$  and any operator  $T : X_t \rightarrow X_s$ . Since  $W$  is separable and  $\Gamma$  is uncountable, there exist  $t \neq s \in \Gamma$  such that  $t(i) = s(i)$  ( $i \leq k$ ) with

$$\|T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})\| \leq 1/\lambda 2^k, \quad j \in I_k.$$

Define now  $T : X_t \rightarrow X_s$  by  $T = T_s^{-1} P_s T_t$ , where  $T_s^{-1} : T_s X_s \rightarrow X_s$ . We have  $T_s^{-1} P_s (T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})) = T(z_{j,t(k)}) - z_{j,s(k)}$ , and therefore

$$\sum_{j \in I_k} \|T z_{j,t(k)} - z_{j,s(k)}\| \leq \sum_{j \in I_k} \|T_s^{-1} P_s\| \|T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})\| \leq 1.$$

Hence we have,

$$\left| \beta_t^k(T) \right| \geq 1 - 2^{-k} \sum_{j \in I_k} \left\| z_{j,s(k)}^* (z_{j,s(k)} - T z_{j,t(k)}) \right\| \geq 1 - 2^{-k}.$$

Now since  $t(m) \neq s(m)$  for some  $m > k$  and  $(z_{j,\epsilon}^*, z_{j,\epsilon})$  is a biorthogonal system, we have

$$\beta_t^m(T) = 0.$$

Therefore,

$$\|T\| \geq \delta^{-1} \left| \beta_t^m(T) - \beta_t^k(T) \right| \geq (1/2)\delta^{-1}.$$

On the other hand,

$$\|T\| \leq \|T_s^{-1}\| \|P_s\| \|T_t\| \leq \lambda^2.$$

Since  $\delta$  was arbitrary, we get a contradiction.  $\square$

**Theorem 6.** *Every separable Banach space satisfying the Cantorized-Enflo criterion is ergodic.*

*Proof.* Let  $X$  be a separable Banach space satisfying the Cantorized-Enflo criterion. Define an equivalence relation  $E$  on  $2^{\mathbb{N}}$  by setting  $sEt$  if and only if  $X_s$  is isomorphic to  $X_t$ . We observe that  $E$  is  $E'_0$ -invariant. Indeed, if  $tE'_0s$  then  $X_t$  and  $X_s$  are generated by the same sequence of vectors except for finite sets of the same cardinality, and therefore are isomorphic spaces. By Lemma 5 each equivalence class of  $E$  is countable and then a meagre subset of  $2^{\mathbb{N}}$ . It is a general fact that an equivalence relation is meagre whenever each of its equivalence class is meagre [29]. Hence  $E$  is a meagre equivalence relation on  $2^{\mathbb{N}}$ , and we have from Proposition 4 that  $E_0 \leq_B E$ . It is clear that the function  $\phi : 2^{\mathbb{N}} \rightarrow \mathcal{SB}(X)$  given by  $\phi(t) = X_t$  is Borel. In consequence,  $X$  is ergodic.  $\square$

**Remark 7.** *A Banach space satisfying the Cantorized-Enflo criterion has a continuum of non isomorphic subspaces failing the bounded approximation property.*

*Proof.* We observe that the spaces  $X_t$  used in the reduction fails the BAP for every  $t \in 2^{\mathbb{N}}$ . Given  $\lambda > 0$ , let  $n \in \mathbb{N}$  be such that  $\lambda \sum_{k>n} \alpha_k \leq 1/2$ . Let  $T : X_t \rightarrow X_t$  be an operator with  $\|T\| \leq \lambda$ . Since  $|\beta_t^n(U)| \leq \|U|_{Z_n}\|$  for every  $U : X_t \rightarrow Z$ , where  $Z_n = \{z_{i,t(n)}, i \in I_n\}$  is a compact set, we have

$$\|(Id-T)|_{Z_n}\| \geq |\beta_t^n(Id-T)| \geq 1 - |\beta_t^n(T)| \geq 1 - \sum_{k>n} |\beta_t^k(T) - \beta_t^{k-1}(T)| \geq 1 - \|T\| \sum_{k>n} \alpha_k > 1/2.$$

$\square$

**Remark 8.** *Actually, if in Definition 3, we have for every  $n \in \mathbb{N}$ ,  $|\beta_t^n(T) - \beta_t^{n-1}(T)| \leq \sup\{\|Tz\|, z \in F_n\}$  for a finite set  $F_n$  of vectors in  $X$ , such that  $\sum_n \sup\{\|z\|, z \in F_n\} < \infty$ , then every  $X_t$  fails the AP ([39, Proposition 1]).*

**Remark 9.** *We also proved that  $E_0$  is Borel reducible to the relation of complemented biembeddability between the subspaces of a separable Banach space satisfying the Cantorized-Enflo criterion.*

Let  $X$  and  $Y$  be two Banach spaces and a constant  $K > 0$ . Recall that  $X$  is said to be  $K$ -crudely finitely representable in  $Y$  if for every finite dimensional subspace  $F$  of  $X$  there exist a linear isomorphism  $T : F \rightarrow T(F) \subseteq Y$  so that  $\|T\|\|T^{-1}\| \leq K$ .  $X$  is said to be finitely representable in  $Y$  if  $X$  is  $(1 + \epsilon)$ -crudely finitely representable in  $Y$  for every  $\epsilon > 0$ . A classical result of Maurey and Pisier [34] states that  $\ell_p(X)$  and  $\ell_q(X)$  are finitely representable in  $X$ , for any Banach space  $X$ . The following Remark is stated in the classical book [33].

**Remark 10.** *It follows from the proof of [32, Theorem 1.a.5] that if  $\ell_p$  is  $K$ -crudely finitely representable in  $Y$ , for some  $1 \leq p \leq \infty$ , then  $Y$  has a subspace  $X$  which has a Schauder decomposition into  $\{X_n\}_{n=1}^\infty$  with  $d(X_n, \ell_p^n) \leq K + 1$  for every  $n \in \mathbb{N}$ .*

**Proposition 11.** *If  $\ell_p$  is crudely finitely representable in a Banach space  $X$  for some  $p > 2$ , then  $X$  satisfies the Cantorized-Enflo criterion.*

*Proof.* The proof of Johnson and Szankowski [25, Section IV] that there does not exist separable Banach space which is complementably universal for the class of subspaces of  $\ell_p$  ( $2 < p < \infty$ ) is by modifying Davie's construction of a subspace of  $\ell_p$  ( $2 < p < \infty$ ) failing AP. We observe that the Johnson and Szankowski construction yields that  $\ell_p$  ( $2 < p < \infty$ ) satisfies the Cantorized-Enflo criterion.

Indeed, fix  $p > 2$ . For every  $n \in \mathbb{N}$ , we denote by  $(f_j^n)_{j=1}^{3 \cdot 2^n}$  the unit vector basis of  $\ell_p^{3 \cdot 2^n}$ . Using the notation of [25, 10], let for  $j \in I_n$  and  $\epsilon = 0, 1$

$$\begin{aligned} z_{j,\epsilon} &= e_{j+\epsilon 2^n}^{n+1}, \\ z_{j,\epsilon}^* &= \alpha_{j+\epsilon 2^n}^{n+1}, \end{aligned}$$

where the vectors  $e_j^k$  defined in [25] have the form:

$$e_j^k = \sum_{l=1}^{3 \cdot 2^{k-1}} \lambda_j^k(l) f_l^k + \sum_{l=1}^{3 \cdot 2^k} \delta_j^k(l) f_l^k.$$

Also the functionals  $\alpha_j^k$  are linear combinations of the biorthogonal functionals  $(f_l^{k*})_{l=1}^{3 \cdot 2^k}$  of  $\ell_q^{3 \cdot 2^k}$ , satisfying  $\alpha_l^k(e_j^i) = \delta_{ki} \delta_{lj}$ . For any operator  $T : X_t \rightarrow \ell_p$ ,

$$|\beta_t^n(T) - \beta_t^{n-1}(T)| \leq \sup\{\|T\Phi_l^{k,t}\|, l \in F_n\},$$

for some vectors  $\Phi_l^{k,t}$  and a finite set  $F_n$ , where  $\|\Phi_l^{k,t}\| \leq A(n+1)^{1/2} 2^{-n(p-2)/2p}$ , uniformly on  $t$  and  $l$ . Therefore  $\ell_p$  satisfies the Cantorized-Enflo criterion.

Actually, we notice that the previous construction only uses that  $\ell_p$  has a natural Schauder decomposition into  $\{\ell_p^{3 \cdot 2^n}\}_{n=2}^\infty$ . Therefore, if  $\ell_p$  ( $p > 2$ ) is crudely finitely representable in  $X$ , then using the Remark 10, there exist a constant  $K > 0$  and a subspace  $Y$  of  $X$  admitting a Schauder decomposition into  $\{X_n\}_{n=1}^\infty$ , such that  $d(X_n, \ell_p^{3 \cdot 2^n}) \leq K$ . Hence, the analogous construction of vectors  $e_j^k$  and  $\alpha_j^k$  can be done as vectors supported in  $X_{k-1}$  and  $X_k$ .  $\square$

**Corollary 12.** *If  $\ell_p$  ( $p > 2$ ) is crudely finitely representable in  $X$ , then  $X$  is ergodic.*

We observe that the construction of Johnson and Szankowski [25, Section IV] satisfies the Cantorized-Enflo criterion in the form of Remark 8, so each of the  $X_t$  constructed fails the AP.

### 3. CASE $p(X) < 2$

In this section we prove ergodicity for separable Banach spaces such that  $p(X) < 2$ . The particular case for the  $\ell_p$ 's spaces ( $1 \leq p < 2$ ) was proved by Ferenczi and Galego [14], where they actually reduce the relation  $E_{K_\sigma}$  and use only subspaces with unconditional bases. Their approach relies on certain lower estimates on successive vectors which have no reason to hold in the case when  $\ell_p$  is only crudely finitely representable on  $X$ .

Our approach is to obtain the ‘Cantorized version’ of the subspaces of  $\ell_p$  ( $1 \leq p < 2$ ) without AP constructed by Szankowski [39]. The advantage of this method is that the nature of that construction allows to pass the Cantorized-Enflo criterion from  $\ell_p$  to a Banach space  $X$  for which  $\ell_p$  is crudely finitely representable in  $X$ .

Before the proof, we need to define the following functions  $f_k : \mathbb{N} \rightarrow \mathbb{N}$ ,  $k \leq 8$ ,  $g_k : \mathbb{N} \rightarrow \mathbb{N}$ ,  $k \leq 15$ ,  $h_k : \mathbb{N} \rightarrow \mathbb{N}$ ,  $k \leq 32$  to encode the support of some vectors used in that construction. The main difference with [39] is that our construction uses vectors with support of length twelve instead of six of the original one.

$$\begin{aligned} f_k(16i + l) &= 8i + k - 1, \quad i = 2, 3, 4, \dots \quad 0 \leq l \leq 15, \quad 1 \leq k \leq 8 \\ g_k(16i + l) &= 16i + (l + k) \bmod 16, \quad i = 2, 3, 4, \dots \quad 0 \leq l \leq 15, \quad 1 \leq k \leq 15 \\ h_k(16i + l) &= 32i + k - 1, \quad i = 2, 3, 4, \dots \quad 0 \leq l \leq 3, \quad 1 \leq k \leq 32 \end{aligned}$$

We denote by  $I_n^j = \{k \in I_n : k \cong j \pmod{16}\}$ ,  $j = 0, 1, 2, \dots, 15$ . The following is a modified version of the key Szankowski combinatorial argument [39] (See also [33, Prop 1.g.5]) adapted to our set of functions  $\{f_k, g_k, h_k\}$ .

**Lemma 13.** *There exist partitions  $\Delta_n$  and  $\nabla_n$  of  $I_n$  into disjoint sets and a sequence of integers  $(m_n)_n$  with  $m_n \geq 2^{n/32-1}$ ,  $n = 2, 3, \dots$  such that*

- (1) *For every  $A \in \nabla_n$ ,  $m_n \leq |A| \leq 2m_n$  and it is contained in some  $I_n^j$ .*
- (2) *For every  $A \in \nabla_n$  and every  $B \in \Delta_n$ ,  $|A \cap B| \leq 1$ .*
- (3) *For every  $A \in \nabla_n$  and every function  $\xi$  in  $\{f_k, g_k, h_k\}$ , the set  $\xi(A)$  is contained entirely in an element of  $\Delta_{n-1}, \Delta_n$  or  $\Delta_{n+1}$ .*

*Proof.* Consider the functions  $\varphi_n^j : I_n^0 \rightarrow I_n^j$  given by  $\varphi_n^j(k) = k + j$ . For  $n \geq 4$  and  $r = 0, 1$  we let  $\psi_n^r : I_n^0 \rightarrow I_{n+1}^0$  the map defined by  $\psi_n^r(k) = 2k + 16r$ . The above functions are 1-1 and have disjoint ranks with  $I_{n+1}^0 = \psi_n^0(I_n^0) \cup \psi_n^1(I_n^0)$ .

Inductively, for  $n \geq 4$  we can represent  $I_n^0$  as the cartesian product  $C_n \times D_n$ , where  $|D_{n+1}| = |C_n|$ ,  $|C_{n+1}| = 2|D_n|$  and such that

- (1) For every  $c \in C_{n+1}$  there exist  $d \in D_n$  and  $r = 0, 1$  such that  $\psi_n^r(C_n \times \{d\}) = \{c\} \times D_{n+1}$ .
- (2) For every  $d \in D_{n+1}$  there exists  $c \in C_n$  such that  $\psi_n^0 \cup \psi_n^1(\{c\} \times D_n) = C_{n+1} \times \{d\}$ .



This means that the functions  $\psi_n^r$  send columns of  $C_n \times D_n$  onto rows of  $C_{n+1} \times D_{n+1}$  in a way that every column of  $C_{n+1} \times D_{n+1}$  is the image of a row of  $C_n \times D_n$  by  $\psi_n^0 \cup \psi_n^1$ . Notice that  $|C_n|, |D_n| \geq 2^{n/2-2}$ .

Now we split each  $D_n$  as a cartesian product of sixteen factors  $D_n = \prod_{l=0}^{15} D_n^l$  such that

$$|D_n^0| \leq |D_n^1| \leq \dots \leq |D_n^{15}| \leq 2|D_n^0|$$

The partitions are then defined as

$$\begin{aligned} \nabla_n &= \left\{ \varphi_n^l \left( \{c\} \times D_n^l \right) : c \in C_n \times \prod_{i \neq l} D_n^i, 0 \leq l \leq 15 \right\}, \\ \Delta_n &= \left\{ \varphi_n^l \left( C_n \times \prod_{i \neq l} D_n^i \times \{d\} \right) : d \in D_n^l, 0 \leq l \leq 15 \right\}. \end{aligned}$$

The conditions 1), 2) and 3) are satisfied in the same way as [39] □

**Theorem 14.** *If  $\ell_p$  is crudely finitely representable in a Banach space  $X$ , for some  $1 \leq p < 2$ , then  $X$  satisfies the Cantorized-Enflo criterion.*

*Proof.* Let  $X$  be a Banach space such that  $\ell_p$  is crudely finitely representable, for some  $1 \leq p < 2$ . For every  $n \in \mathbb{N}$ , we fix  $\Delta_n$  and  $\nabla_n$  partitions of  $I_n$  obtained by Lemma 13. It follows by Remark 10 that there exist a constant  $K > 0$  and a subspace  $Y$  of  $X$  admitting a Schauder decomposition into  $\{X_n\}_{n=1}^\infty$  such that  $d(X_n, \ell_p^{2^n}) \leq K$ , for every  $n \in \mathbb{N}$ . Let  $(x_j)_{j=1}^\infty$  be a bounded sequence of vectors in  $Y$  with  $x_j \in X_n$  when  $j \in I_n$  such that for every  $n$

$$(1) \quad K^{-1} \left( \sum_{B \in \Delta_n} \left( \sum_{j \in B} |a_j|^2 \right)^{p/2} \right)^{1/p} \leq \left\| \sum_{j \in I_n} a_j x_j \right\| \leq K \left( \sum_{B \in \Delta_n} \left( \sum_{j \in B} |a_j|^2 \right)^{p/2} \right)^{1/p},$$

for any sequence of scalars  $(a_j)_{j=1}^\infty$ . Let  $(x_j^*)_{j=1}^\infty$  be a sequence of functionals in  $Y^*$ , such that  $x_j^*(x_i) = \delta_{ij}$  for all  $i, j \in \mathbb{N}$  and

$$(2) \quad K^{-1} \left( \sum_{B \in \Delta_n} \left( \sum_{j \in B} |b_j|^2 \right)^{q/2} \right)^{1/q} \leq \left\| \sum_{j \in I_n} b_j x_j^* \right\|_{Y^*} \leq K \left( \sum_{B \in \Delta_n} \left( \sum_{j \in B} |b_j|^2 \right)^{q/2} \right)^{1/q}$$

for any sequence of scalars  $(b_j)_{j=1}^\infty$  and every  $n$ , where  $1/p + 1/q = 1$ .

We now define the sequence of vectors  $(z_{i,\epsilon})_i$ ,  $\epsilon = 0, 1$  in  $Y$  by setting:

$$z_{i,0} = (x_{8i} - x_{8i+1}) + (x_{8i+2} - x_{8i+3}) + x_{16i} + x_{16i+1} + x_{16i+4} + x_{16i+5} + x_{16i+8} + x_{16i+9} + x_{16i+12} + x_{16i+13}$$

$$z_{i,1} = (x_{8i+4} - x_{8i+5}) + (x_{8i+6} - x_{8i+7}) + x_{16i+2} + x_{16i+3} + x_{16i+6} + x_{16i+7} + x_{16i+10} + x_{16i+11} + x_{16i+14} + x_{16i+15}$$

Recall that  $Z = \overline{\text{span}}\{z_{j,\epsilon} : j \in \mathbb{N}, \epsilon = 0, 1\}$ . Notice that for every  $i \in \mathbb{N}$ ,

$$\begin{aligned} (x_{8i}^* - x_{8i+1}^*)|_Z &= (x_{8i+2}^* - x_{8i+3}^*)|_Z = 1/2(x_{16i}^* + x_{16i+1}^* + x_{16i+8}^* + x_{16i+9}^*)|_Z \\ &= 1/2(x_{16i+4}^* + x_{16i+5}^* + x_{16i+12}^* + x_{16i+13}^*)|_Z. \end{aligned}$$

Indeed, all four formulas give 2 when evaluated on  $z_{i,0}$  and give 0 when evaluated on  $z_{j,\epsilon} \neq z_{i,0}$ . Analogously, for every  $i \in \mathbb{N}$ ,

$$\begin{aligned} (x_{8i+4}^* - x_{8i+5}^*)|_Z &= (x_{8i+6}^* - x_{8i+7}^*)|_Z = 1/2(x_{16i+2}^* + x_{16i+3}^* + x_{16i+10}^* + x_{16i+11}^*)|_Z \\ &= 1/2(x_{16i+6}^* + x_{16i+7}^* + x_{16i+14}^* + x_{16i+15}^*)|_Z. \end{aligned}$$

All four formulas above give 2 when evaluated on  $z_{i,1}$  and 0 when evaluated on  $z_{j,\epsilon} \neq z_{i,1}$ . We define the sequence of functionals  $(z_{n,\epsilon}^*)_{n \in \mathbb{N}}$ ,  $\epsilon = 0, 1$  on  $Z^*$  by setting

$$z_{i,\epsilon}^* = 1/2(x_{8i+4\epsilon}^* - x_{8i+4\epsilon+1}^*)|_Z.$$

Hence,

$$\begin{aligned} z_{i,0}^* &= 1/2(x_{8i+2}^* - x_{8i+3}^*)|_Z = 1/4(x_{16i}^* + x_{16i+1}^* + x_{16i+8}^* + x_{16i+9}^*)|_Z \\ &= 1/4(x_{16i+4}^* + x_{16i+5}^* + x_{16i+12}^* + x_{16i+13}^*)|_Z, \\ z_{i,1}^* &= 1/2(x_{8i+6}^* - x_{8i+7}^*)|_Z = 1/4(x_{16i+2}^* + x_{16i+3}^* + x_{16i+10}^* + x_{16i+11}^*)|_Z \\ &= 1/4(x_{16i+6}^* + x_{16i+7}^* + x_{16i+14}^* + x_{16i+15}^*)|_Z. \end{aligned}$$

For  $t \in 2^{\mathbb{N}}$ , recall that  $X_t = \overline{\text{span}}\{z_{j,t(n)} : j \in I_n, n \in \mathbb{N}\}$ . If  $T : X_t \rightarrow Z$  is a linear and bounded operator, the  $n$ -trace of  $T$  has been defined as

$$\beta_t^n(T) = 2^{-n} \sum_{j \in I_n} z_{j,t(n)}^* T(z_{j,t(n)}).$$

We need to verify that the  $\beta_n^t$ s satisfy the conditions of the Cantorized-Enflo criterion (Definition 3).

CASE 1:  $t(n) = t(n-1) = 0$ .

$$\begin{aligned} \beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,0}^* T(z_{i,0}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,0}^* T(z_{i,0}) \\ &= 2^{-n} \sum_{i \in I_n} 2^{-1} (x_{8i}^* - x_{8i+1}^*) T(z_{i,0}) - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i}^* + x_{16i+1}^* + x_{16i+8}^* + x_{16i+9}^*) T(z_{i,0}) \\ &= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i}^* T(z_{2i,0} - z_{i,0}) + x_{16i+1}^* T(-z_{2i,0} - z_{i,0}) \\ &\quad + x_{16i+8}^* T(z_{2i+1,0} - z_{i,0}) + x_{16i+9}^* T(-z_{2i+1,0} - z_{i,0})\} \end{aligned}$$

The elements in parentheses above will be called  $y_{16i}, y_{16i+1}, y_{16i+8}, y_{16i+9}$  respectively, thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(0,0)} x_j^* T(y_j),$$

where  $I_n(0, 0) = I_n^0 \cup I_n^1 \cup I_n^8 \cup I_n^9$ .

CASE 2:  $t(n) = 0, t(n-1) = 1$

$$\begin{aligned}
\beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,0}^* T(z_{i,0}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,1}^* T(z_{i,1}) \\
&= 2^{-n} \sum_{i \in I_n} 2^{-1} (x_{8i+2}^* - x_{8i+3}^*) T(z_{i,0}) - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i+2}^* + x_{16i+3}^* + x_{16i+10}^* + x_{16i+11}^*) T(z_{i,1}) \\
&= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i+2}^* T(z_{2i,0} - z_{i,1}) + x_{16i+3}^* T(-z_{2i,0} - z_{i,1}) \\
&\quad + x_{16i+10}^* T(z_{2i+1,0} - z_{i,1}) + x_{16i+11}^* T(-z_{2i+1,0} - z_{i,1})\}
\end{aligned}$$

The elements in parentheses above will be called  $y_{16i+2}, y_{16i+3}, y_{16i+10}, y_{16i+11}$  respectively, thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(0,1)} x_j^* T(y_j),$$

where  $I_n(0, 1) = I_n^2 \cup I_n^3 \cup I_n^{10} \cup I_n^{11}$ .

CASE 3:  $t(n) = 1, t(n) = 0$

$$\begin{aligned}
\beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,1}^* T(z_{i,1}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,0}^* T(z_{i,0}) \\
&= 2^{-n} \sum_{i \in I_n} 2^{-1} (x_{8i+4}^* - x_{8i+5}^*) T(z_{i,1}) - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i+4}^* + x_{16i+5}^* + x_{16i+12}^* + x_{16i+13}^*) T(z_{i,0}) \\
&= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i+4}^* T(z_{2i,1} - z_{i,0}) + x_{16i+5}^* T(-z_{2i,1} - z_{i,0}) \\
&\quad + x_{16i+12}^* T(z_{2i+1,1} - z_{i,0}) + x_{16i+13}^* T(-z_{2i+1,1} - z_{i,0})\}
\end{aligned}$$

The elements in parentheses above will be called  $y_{16i+4}, y_{16i+5}, y_{16i+12}, y_{16i+13}$  respectively, thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(1,0)} x_j^* T(y_j),$$

where  $I_n(1, 0) = I_n^4 \cup I_n^5 \cup I_n^{12} \cup I_n^{13}$ .

CASE 4:  $t(n) = 1, t(n-1) = 1$

$$\begin{aligned}
\beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,1}^* T(z_{i,1}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,1}^* T(z_{i,1}) \\
&= 2^{-n} \sum_{i \in I_n} 2^{-1} (x_{8i+6}^* - x_{8i+7}^*) T(z_{i,1}) - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i+6}^* + x_{16i+7}^* + x_{16i+14}^* + x_{16i+15}^*) T(z_{i,1}) \\
&= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i+6}^* T(z_{2i,1} - z_{i,1}) + x_{16i+7}^* T(-z_{2i,1} - z_{i,1}) \\
&\quad + x_{16i+14}^* T(z_{2i+1,1} - z_{i,1}) + x_{16i+15}^* T(-z_{2i+1,1} - z_{i,1})\}
\end{aligned}$$

The elements in parentheses above will be called  $y_{16i+6}, y_{16i+7}, y_{16i+14}, y_{16i+15}$  respectively, thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(1,1)} x_j^* T(y_j),$$

where  $I_n(1,1) = I_n^6 \cup I_n^7 \cup I_n^{14} \cup I_n^{15}$ .

Hence,

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(t(n), t(n-1))} x_j^* T(y_j).$$

We use the functions  $\{f_k, g_k, h_k\}$  to describe the support of the vectors  $y_j$ . For each  $j$ , we shall need four functions of the  $\{f_k, k \leq 8\}$ , nine functions of the  $\{g_k, k \leq 15\}$  and eight functions of the  $\{h_k, k \leq 32\}$ . In fact, notice that

$$y_j = \sum_{k=1}^4 \alpha_{j_k} x_{f_{j_k}(j)} + \sum_{t=1}^9 \beta_{j_t} x_{g_{j_t}(j)} + \sum_{s=1}^8 \gamma_{j_s} x_{h_{j_s}(j)},$$

where  $|\alpha_{j,k}| = |\gamma_{j,s}| = |\beta_{j,t}| = 1$  for all the indexes in the formula above, except for one  $j_{t_0}$  which satisfies  $|\beta_{j_{t_0}}| = 2$ .

Given  $\epsilon, \delta = 0, 1$ , we write  $\nabla_n(\epsilon, \delta) = \{A \in \nabla_n : A \subseteq I_n(\epsilon, \delta)\}$ . Notice that

$$\begin{aligned}
2^{-n-1} \sum_{j \in I_{n+3}(\epsilon, \delta)} x_j^* T(y_j) &= 2^{-n-1} \sum_{A \in \nabla_{n+3}(\epsilon, \delta)} \sum_{j \in A} x_j^* T(y_j) \\
&= 2^{-n-1} \sum_{A \in \nabla_{n+3}(\epsilon, \delta)} 2^{-|A|} \sum_{\theta} \left( \sum_{j \in A} \theta_j x_j^* \right) \left( \sum_{j \in A} \theta_j T y_j \right),
\end{aligned}$$

where the sum is taken over all the choices of signs  $\{\theta_j\}_{j \in A}$ . Observe that by Lemma 13-(2) and the equation (2) above (about norm of the functionals  $x_j^*$ 's) we have, for every  $A \in \nabla_{n+3}(\epsilon, \delta)$  and  $\{\theta_j\}_{j \in A}$ ,

$$\begin{aligned} \left\| \sum_{j \in A} \theta_j x_j^* \right\|_{Y^*} &\leq K \left( \sum_{B \in \Delta_n} \left( \sum_{j \in B \cap A} |\theta_j|^2 \right)^{q/2} \right)^{1/q} \\ &= K|A|^{1/q} \leq K(2m_{n+3})^{1/q}, \end{aligned}$$

where  $1/p + 1/q = 1$ . By Lemma 13-(3) we have for every  $A \in \nabla_{n+3}(\epsilon, \delta)$ ,  $\{\theta_j\}_{j \in A}$  and any function  $\xi$  in  $\{f_k, g_k, h_k\}$

$$\left\| \sum_{j \in A} \theta_j x_{\xi(j)} \right\| \leq K|A|^{1/2} \leq K(2m_{n+3})^{1/2}.$$

It follows that

$$\left\| \sum_{j \in A} y_j \right\| = \left\| \sum_{j \in A} \sum_{k=1}^{21} \lambda_{j_k} x_{\xi_{l_k}(j)} \right\| \leq 42K|A|^{1/2} \leq 42K(2m_{n+3})^{1/2}.$$

Notice that by construction  $|\nabla_n(\epsilon, \delta)| = 2^{-2}|\nabla_n| \leq 2^{n-2}m_n^{-1}$ . Hence,

$$\begin{aligned} |\beta_t^n(T) - \beta_t^{n-1}(T)| &= \left| 2^{-n-1} \sum_{j \in I_{n+3}(\epsilon, \delta)} x_j^* T(y) \right| \\ &\leq 2^{-n-1} (2^{n+1} m_{n+3}^{-1}) K(2m_{n+3})^{1/q} 42K \sqrt{2} m_{n+3}^{1/2} \|T\| \\ &\leq 84K^2 (m_{n+3})^{1/q+1/2-1} \|T\| \end{aligned}$$

Since  $\alpha = 1/2 + 1/q - 1 = 1/2 - 1/p < 0$ , the series  $\sum_n m_n^\alpha \leq \sum_n 2^{\alpha(n/32-1)} < \infty$ . Therefore,  $X$  satisfies the Cantorized-Enflo criterion.  $\square$

**Remark 15.** *The proof of the Theorem 14 is based on the idea from [39] where subspaces of  $\ell_p$  ( $1 \leq p < 2$ ) without AP were constructed. It was pointed out by Szankowski [39] (see also [33, Remark 2. pg 111]) that the mentioned idea can be easily adapted to obtain subspaces of  $\ell_p$  ( $2 < p < \infty$ ) without AP. This implies that also the method used in the proof of Theorem 14 is valid for Banach spaces  $X$  in which  $\ell_p$  ( $2 < p < \infty$ ) is crudely finitely representable. Indeed, the same definition of vectors  $z_{i,\epsilon}$  and functionals  $z_{i,\epsilon}^*$  works, it is only necessary to modify the construction of the partitions  $\Delta_n$  and  $\nabla_n$  in Lemma 13. This gives us an independent proof of Proposition 11.*

We observe that the construction of Theorem 14 satisfies the Cantorized-Enflo criterion in the form of Remark 8, therefore every  $X_t$  constructed above fails the AP. We can conclude that:

**Theorem 16.** *Every separable Banach space non near Hilbert satisfies the Cantorized-Enflo criterion and therefore is ergodic. Furthermore, the reduction uses subspaces without AP.*

The following remark is due to R. Anisca.

**Remark 17.** *There do exist near-Hilbert spaces satisfying the Cantorized-Enflo criterion. Indeed, Casazza, García and Johnson [6] constructed an asymptotically Hilbertian space which fails the AP. Their approach follows closely the Davie construction [10] of a subspace of  $\ell_p = (\sum_n \ell_p^{3 \cdot 2^n})_p$  ( $2 < p < \infty$ ) failing AP. The space in [6] is instead a subspace of  $Z = (\sum_n \ell_{p_n}^{3 \cdot 2^n})_2$  where  $p_n \downarrow 2$  appropriately. One can combine the arguments of Proposition 11 and those in [25, Section IV] to construct a version of the Casazza, García and Johnson space satisfying the Cantorized-Enflo criterion. Also, the arguments from Theorem 16 can be used in the context of construction by Anisca and Chlebovec [3] and obtain that spaces of the form  $\ell_2(X)$ , with  $X$  of cotype 2 and having the sequence of Euclidean distances of order at least  $(\log n)^\beta$  ( $\beta > 1$ ), satisfy the Cantorized-Enflo criterion.*

As a direct consequence of Lemma 5 and Theorem 14, we can now extend the result of Johnson and Szankowski [25] about complementably universal spaces for the family of subspaces of  $\ell_p$  ( $2 < p < \infty$ ).

**Theorem 18.** *There is no separable Banach space which is complementably universal for the class of all subspaces of  $X$  when  $X$  is non near Hilbert.*

**Corollary 19.** *There is no separable Banach space which is complementably universal for the family of subspaces of  $\ell_p$  ( $1 \leq p < 2$ ).*

In the limit case, Johnson and Szankowski [27] constructed a separable space, non isomorphic to the Hilbert, such that all subspaces have the BAP, and which is complementably universal for all its subspaces. Also, if every subspace of  $X$  has BAP (for example if  $X$  is weak Hilbert), then the Pełczyński's universal space (see [32, Theorem 2.d.10(a)]) is complementably universal for the family of all subspaces of  $X$ .

#### 4. THE SCHLUMPRECHT TYPE SPACE $S_{p,r}$ IS NOT NEAR HILBERT

Schlumprecht [1, 40] constructed the first example of a complementably minimal Banach space  $S$  different from the classical spaces  $c_0$  and  $\ell_p$  ( $1 \leq p < \infty$ ). In [8], the Schlumprecht construction was extended to uniformly convex examples using interpolation techniques. In fact, they constructed a family of uniformly convex complementably minimal spaces by interpolating  $S$  and  $\ell_q$ .

The approach in [8] deals with Banach spaces  $X$  defined by lattice norms  $\|\cdot\|_X$  on  $c_{00}$ . In this context, if  $X$  and  $Y$  are two such spaces and  $0 < \theta < 1$ , then  $X^{1-\theta}Y^\theta$  is defined as the space  $Z$  with the norm  $\|z\|_Z = \inf\{\|x\|_X^{1-\theta}\|y\|_Y^\theta, z = |x|^{1-\theta}|y|^\theta\}$ . When we consider the complex scalars and either  $X$  or  $Y$  is separable, then  $Z$  coincides with the usual complex interpolation space  $[X, Y]_\theta$  (see [5]).

**Definition 20.** For every  $1 \leq p < r \leq \infty$ , the Schlumprecht type space  $S_{p,r}$  is defined as the interpolated space  $\ell_t^{1-\theta} S^\theta$ , where  $\theta = \frac{1}{p} - \frac{1}{r}$  and  $t = (1 - \theta)r$ .

**Proposition 21.** [8, Proposition 3 and Theorem 8] For any  $1 \leq p < r \leq \infty$ , the space  $S_{p,r}$  and its dual are complementably minimal. Furthermore,  $S_{p,r}$  has an 1-unconditional normalized basis  $(e_n)_{n=1}^\infty$  such that  $\|\sum_{i=1}^n e_i\|_{S_{p,r}} = n^{1/p} \log_2(n+1)^{1/r-1/p}$ , for every  $n \in \mathbb{N}$ .

Notice that  $S$  is simply  $S_{1,\infty}$ . We use the ideas from [9] and estimates of the norm of some combination of the vector basis to compute  $p(S_{p,r})$  and  $q(S_{p,r})$ .

**Proposition 22.** Let  $1 \leq p < r \leq \infty$ . For every  $n \in \mathbb{N}$  and  $\epsilon > 0$ , there exists a sequence of vectors  $v_1, \dots, v_n$  in  $c_{00}$  such that

- (1) The set of vectors  $\{v_1, \dots, v_n\}$  are disjointly supported.
- (2)  $\|\epsilon_1 v_1 + \dots + \epsilon_n v_n\|_{S_{p,r}} \leq (1 + \epsilon)^\theta n^{1/r}$ , for any  $\epsilon_1, \dots, \epsilon_n$  of modulus 1.

*Proof.* D. Kutzarova and P. K. Lin [31] proved that there exists vectors  $v_1, \dots, v_n$  in  $c_{00}$  which are disjointly supported such that  $\|v_1 + \dots + v_n\|_S \leq (1 + \epsilon)$ , where each  $v_j$  is of the form  $\frac{m_j}{\log_2(m_j+1)} \sum_{i \in M_j} e_i$ ,  $|M_j| = m_j$ . Since the basis of  $S$  is 1-unconditional, we have  $\|\epsilon_1 v_1 + \dots + \epsilon_n v_n\|_S \leq (1 + \epsilon)$  for any  $(\epsilon_j)_j^n$  of modulus 1.

Let  $v = \epsilon_1 v_1 + \dots + \epsilon_n v_n$ , it follows from the Lozanovskii formula that

$$\|v\|_{S_{p,r}} \leq \|v\|_t^{1-\theta} \|v\|_S^\theta \leq (1 + \epsilon)^\theta n^{(1-\theta)/t} = (1 + \epsilon)^\theta n^{1/r}.$$

□

**Proposition 23.** Let  $1 \leq p < r \leq \infty$ . The family of Schlumprecht type spaces  $S_{p,r}$  and their duals are not near Hilbert. In particular, they are ergodic spaces.

*Proof.* Let  $1 \leq p < r \leq \infty$ . Assume that  $S_{p,r}$  has type  $t$  then by Proposition 21,  $n^{1/p} \log_2(n+1)^{1/r-1/p} \leq T_t n^{1/t}$  for some constant  $T_t$ , and then  $t \leq p$ . Hence  $p(S_{p,r}) \leq p$ . Analogously, if  $S_{p,r}$  has cotype  $t$ , then by Proposition 22  $t \geq r$ , and then  $q(S_{p,r}) \geq r$ . We have that  $p(S_{p,r}) \leq p < r \leq q(S_{p,r})$  and it follows that  $S_{p,r}$  is not near Hilbert. Also, since a Banach space  $X$  is near Hilbert if and only if  $X^*$  is near Hilbert, the dual space  $S_{p,r}^*$  is not near Hilbert. □

## 5. FINAL REMARKS

Of course, the main question concerning ergodic spaces is whether  $\ell_2$  is the only non ergodic Banach space. The conclusion of Theorem 16 restricts the question of ergodicity to the case of near Hilbert spaces, but our technique uses a reduction throughout subspaces without AP. A Banach space in which all of its subspaces have AP is said to have the *hereditarily approximation property* (HAP). Szankowski [39] proved that every HAP space must be near Hilbert. The first example of a HAP space not isomorphic to a Hilbert space was constructed by Johnson [22]. Later, Pisier [35] proved that every weak Hilbert space has the HAP. The space constructed by Johnson is asymptotically Hilbertian, and therefore ergodic by the Anisca [2] result. In 2010 Johnson and Szankowski [27] constructed a HAP

space with a symmetric basis but not isomorphic to  $\ell_2$ , and hence not asymptotically Hilbertian. Hence, a natural question is the following:

**Problem.** Is the HAP non asymptotically Hilbertian space constructed in [27] ergodic?

Or more generally:

**Problem.** Is every HAP not isomorphic to the Hilbert space ergodic?

Another interesting class of near Hilbert spaces are the twisted Hilbert spaces. The most important example of non trivial twisted Hilbert space is the Kalton-Peck space  $Z_2$  [28].  $Z_2$  is not asymptotically Hilbertian and it is not known whether has HAP.

**Problem.** Does there exist an ergodic non trivial twisted Hilbert space?

Another natural question is:

**Problem.** Is every minimal Banach space not isomorphic to a Hilbert space ergodic?

## 6. ACKNOWLEDGMENTS

The author thanks Valentin Ferenczi for his helpful discussions and comments on this paper; Razvan Anisca and Bill Johnson for their remarks and suggestions.

## REFERENCES

- [1] G. Androulakis and T. Schlumprecht, *The Banach space  $S$  is complementably minimal and subsequentially prime*, Studia Math. 156 (2003), 227–242.
- [2] R. Anisca, *The ergodicity of weak Hilbert spaces*, Proc. of the Ame. Math. Soc. 138 (4) (2010), 1405–1413.
- [3] R. Anisca and C. Chlebovec, *Subspaces of  $\ell_2(X)$  without the approximation property* J. Math. Anal. Appl. 395 (2012), 523–530.
- [4] B. Bossard, *A coding of separable Banach spaces. Analytic and co-analytic families of Banach spaces*, Fund. Math. 172 (2002), 117–152.
- [5] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.
- [6] P. G. Casazza, C. L. Garcia and W. B. Johnson, *An example of an asymptotically Hilbertian space which fails the approximation property*, Proc. Amer. Math. Soc. 129 (2001), 3017–3023.
- [7] P. G. Casazza, W. B. Johnson and L. Tzafriri, *On Tsirelson’s space*, Israel J. Math. 47 (1984), 81–98.
- [8] P. G. Casazza, N. J. Kalton, D. Kutzarova and M. Mastyło, *Complex interpolation and complementably minimal spaces*, Lecture Notes in Pure and Appl. Math. 175, Dekker, New York, 1996, 135–143.
- [9] J. M. F. Castillo, V. Ferenczi and Y. Moreno, *On Uniformly Finitely Extensible Banach spaces*, J. Math. Anal. Appl. 410 (2014) 670–686.
- [10] A. M. Davie, *The approximation problem for Banach spaces*, Bull. London Math. Soc. 5 (1973), 261–266.
- [11] P. Enflo, *A counterexample to the approximation property*, Acta Math. 13 (1973), 309–317.
- [12] S. Dilworth, V. Ferenczi, D. Kutzarova and E. Odell, *On strongly asymptotic  $l_p$  spaces and minimality*, J. Lond. Math. Soc.(2) 75 (2007), 409–419.



- [13] V. Ferenczi, *Minimal subspaces and isomorphically homogeneous sequences in a Banach space*, Israel J. Math. 156 (2006), 125–140.
- [14] V. Ferenczi and E. Galego, *Some equivalence relations which are Borel reducible to isomorphism between separable Banach spaces*, Israel J. of Math. 152 (2006) 61–82.
- [15] V. Ferenczi, A. Louveau, and C. Rosendal, *The complexity of classifying separable Banach spaces up to isomorphism*, J. Lond. Math. Soc. (2) 79 (2009), 323–345.
- [16] V. Ferenczi and C. Rosendal, *Ergodic Banach spaces*, Advances in Mathematics 195 (2005), 259–282.
- [17] V. Ferenczi and C. Rosendal, *On the number of non isomorphic subspaces of a Banach space*, Studia Mathematica 168 (2005), 203–216.
- [18] T. Figiel, *Factorization of compact operators and applications to the approximation problem*, Studia Math. 45 (1973) 191–210.
- [19] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. of Symbolic Logic 54 (1989), 894–914.
- [20] W. T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. 156 (2) (2002), 797–833.
- [21] L. A. Harrington, A. S. Kechris and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. 3 (1990), 903–928.
- [22] W. B. Johnson, *Banach spaces all of whose subspaces have the approximation property*, (Proc. Sem., Ges. Math. Datenverarb., Bonn, 1979), North-Holland, Amsterdam-New York, (1980), 15–26.
- [23] W. B. Johnson, *personal communication*
- [24] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. 6 (1996), 430–470.
- [25] W. B. Johnson and A. Szankowski, *Complementably universal Banach spaces*, Studia Math. 58 (1976), 91–97.
- [26] W. B. Johnson and A. Szankowski, *Complementably universal Banach spaces II*, J. Funct. Anal. 257 (11) (2009), 3395–3408.
- [27] W. B. Johnson and A. Szankowski, *Hereditary approximation property*, Ann. of Math. 176 (2012), 1987–2001.
- [28] N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.
- [29] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1995.
- [30] R. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. 89 (1995), 205–226.
- [31] D. Kutzarova and P. K. Lin, *Remarks about Schlumprecht space*, Proc. Amer. Math. Soc. 128 (2000), 2059–2068.
- [32] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence spaces*, Springer-Verlag, Berlin (1977).
- [33] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function spaces*, Springer-Verlag, Berlin (1979).
- [34] B. Maurey and G. Pisier, *Series of independent, vector valued random variables and geometric properties of Banach spaces* Studia Math. 58 (1976), 45–90.
- [35] G. Pisier, *Weak Hilbert spaces*, Proc. London Math. Soc. 56 (1988), 547–579.
- [36] C. Rosendal, *“Etude Descriptive de l’Isomorphisme dans la Classe des Espaces de Banach”*, These de Doctorat de l’Universite Paris 6, 2003.
- [37] C. Rosendal, *Incomparable, non-isomorphic and minimal Banach spaces*, Fund. Math. 183 (2004), 253–274.
- [38] J.H.Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic 18 (1980), no. 1, 1–28.
- [39] A. Szankowski, *Subspaces without the approximation property*, Israel J. of Math. 30 (1978), 123–129.
- [40] T. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. 76, 1991, 81–95.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, R. DO MATÃO 1010 SP-  
BRAZIL, EMAIL ADDRESS: `CUELLAR@IME.USP.BR`